Exercise 55

Use mathematical induction (page 72) to show that if $f(x) = xe^x$, then $f^{(n)}(x) = (x+n)e^x$.

Solution

Show that the given formula is true for the base case $n = 0$.

$$f^{(0)}(x) = (x+0)e^x$$

$$f(x) = xe^x$$

Now assume that the inductive hypothesis is true for a certain integer k,

$$f^{(k)}(x) = (x+k)e^x, \tag{1}$$

and show that the same formula is true for the consecutive integer $k+1$,

$$f^{(k+1)}(x) = [x + (k+1)]e^x.$$

Take the derivative of both sides of equation (1) with respect to x.

$$\frac{d}{dx}[f^{(k)}(x)] = \frac{d}{dx}[(x+k)e^x]$$

$$\frac{d}{dx}\left(\frac{d^k f}{dx^k}\right) = \left[\frac{d}{dx}(x+k)\right]e^x + (x+k)\left[\frac{d}{dx}(e^x)\right]$$

$$\frac{d^{k+1} f}{dx^{k+1}} = (1)e^x + (x+k)(e^x)$$

$$f^{(k+1)}(x) = e^x + xe^x + ke^x$$

$$= (1+x+k)e^x$$

$$= [x+(k+1)]e^x$$

Therefore, by the principle of mathematical induction,

$$f^{(n)}(x) = (x+n)e^x.$$